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Linear Equations and Inequalities on Finite Dimensional, Real or Complex, Vector Spaces: A Unified Theory*

ADI BEN-ISRAEL

Department of Engineering Sciences, Northwestern University, Evanston, Ill. 60201

Submitted by N. Levinson

INTRODUCTION

Consider the following two theorems:[1]

THEOREM 1. *Let $A \in C^{m \times n}$, $b \in C^m$. Then the system*

$$Ax = b$$

is consistent iff

$$A^H y = 0 \Rightarrow (b, y) = 0. \qquad \square$$

THEOREM 2. *Let $A \in R^{m \times n}$, $b \in R^m$. Then the system*

$$Ax = b, \qquad x \geqq 0$$

is consistent iff

$$A^T y \geqq 0 \Rightarrow (b, y) \geqslant 0. \qquad \square$$

Theorem 1, recurrent throughout linear analysis, is a solvability theorem for linear equations. Theorem 2, the celebrated Farkas theorem [16], is a solvability theorem for linear inequalities. The similarity between these two theorems is not incidental (indeed both are corollaries of Theorem 3.5 below) yet linear equations and linear inequalities are often treated separately and by different methods, as if unrelated. Furthermore, with few exceptions, e.g., [14] and [30], most treatments of linear inequalities (certainly all at undergraduate level) are restricted to the real case.

The purpose of this paper is to present a unified theory of linear equations and inequalities, real or complex. Our presentation is elementary, at under-

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[1] If necessary, consult the notations in Section 0.1.

graduate upper-class level, hence the restriction to finite-dimensional spaces. Our methods are analytic and, to a lesser degree algebraic,[2] but not combinatorial.[3]

All the results are stated for complex spaces, and are readily specialized to the real case, e.g., by ignoring a "Re" in front of a number.

The paper has 5 sections:

Notations and preliminaries are given in Section 0. The prerequisites in analysis are met by a decent undergraduate advanced calculus course. Topological preliminaries, cited without proof in Section 0.2, use metric and finite-dimensional statements and lead to the infimum of a real continuous function on a compact set being attained. This is used in proving the geometric preliminaries of Section 0.3.

Closed convex cones and polar sets are studied in Section 1.

Section 2 gives the solvability theorems for the system

$$Ax = b, \qquad x \in S, \tag{1}$$

where $A \in C^{m \times n}$, $b \in C^m$ and S is a closed convex cone in C^n. Theorems 2.4 and 2.2 characterize the consistency and asymptotic consistency, respectively, of (1). The stronger theorem, 2.4, requires the cone: $N(A) + S$ to be closed, for which it is sufficient that S be a polyhedral convex cone.

Polyhedral systems, i.e., (1) where S is a polyhedral convex cone, are then studied in Section 3. The main theorem, 3.5, states that a polyhedral system (1) is consistent iff

$$A^H y \in S^* \Rightarrow \operatorname{Re}(b, y) \geqslant 0,$$

where S^* is the polar of S. It includes as special cases the above theorems 1 and 2, and suggests extensions of many theorems on linear inequalities to the complex case.[4]

The duality theory of complex linear programming given in Section 4, follows, and slightly extends, Levinson's duality theory given in his fundamental paper [30]. Our proof of the duality theorem, seems new and simple, even in the real case.

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[2] The following quote from [25] is perhaps worth mentioning here: "The student who first meets the ideas of the theory of linear inequalities in, say, a course in linear programming is sometimes perturbed that his recently acquired proficiency in calculus and analysis generally seems to play almost no role. In fact, apart from the theorem that a continuous function defined on a compact set attains its minimum (used in some proofs of the existence of separating hyperplanes), he will probably see no use of analysis."

[3] In algorithmic and other applications of linear inequalities it would of course be necessary to study combinatorial objects such as the face structure of convex polyhedra, vertices and their characterizations, etc.

[4] Such extensions are of course possible by interpreting C^n as R^{2n}, e.g., [30, p. 45 top].

0. NOTATIONS AND PRELIMINARIES

0.1. *Notations*

iff — *if and only if*

F — a *field*, here either

R — the *real field* or

C — the *complex field*

F^n — the *n-dimensional vector space over* F

$F^{m\times n}$ — the $m \times n$ *matrices over* F.

The notation $A \in F^{m\times n}$ is used here loosely, denoting either the matrix A or the linear transformation, of F^n into F^m, represented by A (with respect to a given pair of bases in F^n, F^m).

For any $A = (a_{ij}) \in C^{m\times n}$:

$A^C = (\bar{a}_{ij})$ — the *conjugate*,

A^T — the *transpose*,

$A^H = A^{CT}$ the *conjugate transpose*,

$R(A) = AC^n$ — the *range*,

$N(A) = \{x \in C^n : Ax = 0\}$ — the *null space*,

A^+ — the *generalized inverse* [35], of A.

$A^+ \in C^{n\times m}$ is defined by:

$$A^+y = x \quad \text{if} \quad y = Ax, \quad x \in R(A^H)$$
$$A^+y = 0 \quad \text{if} \quad y \in N(A^H), \quad \text{e.g., [11], [3].}$$

For any $x = (x_i), y = (y_i)$ in C^n :

$(x, y) = y^H x$ — the *(standard) inner product* of x, y

$\| x \| = (x, x)^{1/2}$ — the *(Euclidean) norm* of x

$| x | = (| x_i |) \in R^n$.

For any subspace L in C^n :

$L^\perp = \{y \in C^n : (y, L) = 0\}$ — the *orthogonal complement of* L

P_L — the *(perpendicular) projection* on L

(i.e. $P_L \in C^{n\times n}, P_L = P_L^2 = P_L^H, L = R(P_L)$).

Any result given here for $F = C$ is naturally specialized for $F = R$ (e.g., by reading A^C and A^H as A and A^T, respectively). Some special notations for R^n are:

$R_+^n = \{x \in R^n : x_i \geqslant 0 (i = 1,..., n)\}$ the *nonnegative orthant of* R^n

For any $x, y \in R^n$:

$$x \geqq y \text{ denotes } x - y \in R_+{}^n.$$

Shorthand notations are used throughout, e.g., for any sets S, T in F^n, Λ in F and any functions $f : F^n \to F^m$, $g : F \to F$:

$$\begin{aligned} S + T &= \{s + t : s \in S, t \in T\} \\ (S, T) &= \{(s, t) : s \in S, t \in T\} \\ \Lambda S &= \{\lambda s : \lambda \in \Lambda, s \in S\} \\ f(S) &= \{f(s) : s \in S\} \end{aligned}$$

For any $x = (x_i) \in F^n$:

$$g(x) = (g(x_i)) \in F^n.$$

0.2. *Topological Preliminaries*

Proofs, details and examples of results cited here can be found in most advanced calculus or introductory topology and modern analysis textbooks.

A sequence $\{x_k : k = 1, 2,...\}$ of vectors in C^n *converges* to a vector x, denoted by $x_k \to x$ or $\lim x_k = x$, if $\lim_{k\to\infty} \| x_k - x \| = 0$, in which case x is a *limit point* of any set S in C^n containing (all but a finite number of) $\{x_k : k = 1, 2,...\}$.

A set $S \subset C^n$ is *closed* if it contains all its limit points, and *open* if its complement: comp $S = \{x \in C^n : x \notin S\}$ is closed. Both C^n and $\varnothing$, the empty set, are closed and open.

The *closure* of S, denoted by $c\ell S$, is the smallest closed set containing S (smallest here means that it is contained in any closed set containing S). Equivalently, $c\ell S$ is the intersection of all closed sets containing S. Also, $c\ell S$ is the union of S and all its limit points.

For any $x_0 \in C^n$, $r > 0$, the *ball with center x_0 and radius r* is

$$B(x_0 , r) = \{x \in C^n : \| x - x_0 \| \leqslant r\}.$$

A function $f : C^n \to C^m$ is *continuous at $x_0 \in C^n$* if for any $\epsilon > 0$ there is a $\delta > 0$ such that:

$$x \in B(x_0 , \delta) \Rightarrow f(x) \in B(f(x_0), \epsilon)$$

f is *continuous on a set S* if it is continuous at each point of S.

A set $S \subset C^n$ is *bounded* if $S \subset B(0, r)$ for some $r > 0$.

A bounded sequence $\{x_k : k = 1, 2,...\} \subset C^n$, contains a convergent subsequence.

A set $S \subset C^n$ is *compact* if closed and bounded.

A function $f : C^n \to R$, continuous on a compact set $S \subset C^n$, attains its infimum over S; i.e., there is a point $x_0 \in S$ such that

$$f(x_0) \leqslant f(S)$$

0.3. *Geometric Preliminaries*

0.3.1. DEFINITIONS. A set $S \subset C^n$ is:

(i) *convex* if $0 \leqslant \lambda \leqslant 1 \Rightarrow \lambda S + (1 - \lambda) S \subset S$

(ii) a *cone* if $0 \leqslant \lambda \Rightarrow \lambda S \subset S$

(iii) a *convex cone* if (i) and (ii).

Equivalently, S is a convex cone if (ii) and $S + S \subset S$.

For any $0 \neq u \in C^n$, $\alpha \in R$ the *hyperplane* $H(u, \alpha)$ is

$$H(u, \alpha) = \{x \in C^n : \operatorname{Re}(u, x) = \alpha\}.$$

Note that $H(u, 0)$ is not a subspace of C^n since:

$$x \in H(u, 0), \lambda \in C \not\Rightarrow \lambda x \in H(u, 0).$$

For any set $S \subset C^n$ and a point $b \in C^n$ the *distance* between b and S is: $d(b, S) = \inf\{\| b - x \| : x \in S\}$.

0.3.2. LEMMA. *Let S be a nonempty closed set in C^n and b any point in C^n. Then there is a point $b(S)$ in S which is closest to b, i.e.,*

$$\| b - b(S)\| = d(b, S).$$

Furthermore, if S is convex then $b(S)$ is uniquely determined by b.

PROOF. Assume $b \notin S$, since for $b \in S$ the theorem is obvious with $b(S) = b$. Now $d(b, S) > 0$ since S is closed. For any $z \in S$, the set

$$T = S \cap B(b, \| z - b \|)$$

is closed (intersection of finitely many closed sets) and bounded (since $B(b, \| z - b \|)$ is), hence compact. The continuous function $\| b - x \|$ attains its infimum on the compact set T, say at a point $b(S) \in T$. $b(S)$ is closest to b in T, hence in S.

If S is convex and there are two points u, v in S such that $u \neq v$ and

$$\| b - u \| = \| b - v \| = d(b, S)$$

then

$$\frac{u + v}{2} \in S$$

and

$$\left\| b - \frac{u + v}{2} \right\| < \| b - u \| \qquad \text{(why ?)}$$

contradicting: u closest to b in S. □

0.3.3. LEMMA. *Let $S \subset C^n$ be a convex set, and let b, c be two points in $C^n : b \notin S$, $c \in S$. Then the following are equivalent:*

(a) *c is closest to b in S.*

(b) $x \in S \Rightarrow \operatorname{Re}(c - b, x - c) \geqslant 0$.

PROOF (adapted from [21] p. 99).
(a) $\Rightarrow$ (b): For any $0 \leqslant \lambda \leqslant 1$ and $x \in S$:

$$c + \lambda(x - c) \in S.$$

Now

$$\begin{aligned} 0 &\leqslant \| b - c - \lambda(x - c)\|^2 - \| b - c \|^2 && \text{(by (a))} \\ &= 2\lambda \operatorname{Re}(c - b, x - c) + \lambda^2 \| x - c \|^2 \\ &\quad < 0 \text{ if } \operatorname{Re}(c - b, x - c) < 0 \text{ and } 0 < \lambda < -\frac{2 \operatorname{Re}(c - b, x - c)}{\| x - c \|^2}, \end{aligned}$$

a contradiction.
(b) $\Rightarrow$ (a): For any $x \in S$:

$$\begin{aligned} \| b - x \|^2 - \| b - c \|^2 &= \| x \|^2 - 2 \operatorname{Re}(x, b) + 2 \operatorname{Re}(c, b) - \| c \|^2 \\ &= \| x - c \|^2 + 2 \operatorname{Re}(c - b, x - c) \\ &\geqslant 0, \quad \text{if} \quad (b). \end{aligned}$$

□

0.3.4. THEOREM. *Let S be a nonempty closed convex set in C^n, b a point not in S. Then there is a nonzero $u \in C^n$ and a real α such that*

$$\operatorname{Re}(u, S) \geqslant \alpha$$
$$\operatorname{Re}(u, b) < \alpha.$$

PROOF (adapted from [24]). Let $b(S)$ be the point in S closest to b, see 0.3.2. For any $x \in S$:

$$\begin{aligned} \operatorname{Re}(b(S) - b, x) &\geqslant \operatorname{Re}(b(S) - b, b(S)), \qquad \text{by} \quad 0.3.3, \\ &> \operatorname{Re}(b(S) - b, b), \end{aligned}$$

since

$$\operatorname{Re}(b(S) - b, b(S) - b) = \| b(S) - b \|^2 > 0.$$

The theorem thus holds with

$$u = b(S) - b, \qquad \alpha = \operatorname{Re}(b(S) - b, b(S)).$$

□

Theorem 0.3.4 states that any closed convex set in C^n and a point outside it can be separated by a hyperplane. It is an elementary geometric version of the celebrated Hahn-Banach theorem, e.g., [33] Chapter III, and [12].

0.3.5. COROLLARY. *Let S be a closed convex cone in C^n, $b \notin S$. Then there is a nonzero $u \in C^n$ such that:*

$$\operatorname{Re}(u, S) \geqslant 0$$
$$\operatorname{Re}(u, b) < 0.$$

PROOF. Since $0 \in S$, $\alpha \leqslant 0$ in 0.3.4.
Now $\alpha < 0$ is impossible. (Why?) □

For a comprehensive study of convex sets in normed linear spaces the reader is referred to [15], and the bibliography therein.

1. CLOSED CONVEX CONES AND POLAR SETS

1.1. DEFINITION. Let S be a nonempty set in C^n. The *polar* of S, denoted S^*, is:

$$S^* = \{y \in C^n : \operatorname{Re}(y, S) \geqslant 0\}.$$

1.2. EXAMPLES.

(a) If S is a subspace of C^n then $S^* = S^\perp$.

(b) R, as a subset of C, has the polar: $R^* = iR$

(c) If $S_j \subset C^{n(j)}(j = 1,\dots, k)$ and $S = S_1 \times S_2 \times \cdots \times S_k \subset C^{\Sigma n(j)}$, then

$$S^* = S_1^* \times S_2^* \times \cdots \times S_k^* .$$

(d) Let $0 \leqslant \alpha \leqslant \pi/2$ and let $T_\alpha = \{z \in C : |\arg z| \leqslant \alpha\}$. Then

$$(T_\alpha)^* = \left\{w \in C : |\arg w| \leqslant \frac{\pi}{2} - \alpha\right\} = T_{\pi/2-\alpha}$$

(e) Let $\alpha \in R^n$ satisfy $0 \leqq \alpha \leqq (\pi/2)e$, where $e^T = (1, 1,\dots, 1)$, and let

$$T_\alpha = \{z \in C^n : |\arg z| \leqq \alpha\}.$$

Then

$$(T_\alpha)^* = \left\{w \in C^n : |\arg w| \leqq \frac{\pi}{2} e - \alpha\right\} = T_{(\pi/2)e-\alpha} .$$

1.3. THEOREM. *Let S, T be any sets in C^n. Then:*

(a) *S^* is a closed convex cone*

(b) $S \subset T \Rightarrow T^* \subset S^*$

(c) $S \subset S^{**}$ [5]

[5] $S^{**} = (S^*)^*$, the polar of the polar of S.

(d) $S^* = S^{***}$

(e) $S^* = (c\ell S)^*$

(f) $S^* \cap T^* \subset (S + T)^*$

$(S + T)^* \subset S^* \cap T^* \qquad \text{if} \quad 0 \in S \cap T$

(g) $S^* + T^* \subset (S \cap T)^*$.

PROOFS.

(a), (b), and (c) are easy consequences of Definition 1.1.

(d) $S^* \subset (S^*)^{**}$, by (c)

Conversely

$$(S^{**})^* \subset S^* \text{ follows from (c) and (b).}$$

(e) $(c\ell S)^* \subset S^*$ from $S \subset c\ell S$ and (b).

Conversely, let $x \in S^*$, $\{s_k : k = 1, 2, \ldots\} \subset S$, $s = \lim_{k\to\infty} s_k$. Then

$$\begin{aligned} Re(x, s) &= \lim_{k\to\infty} Re(x, s_k) \\ &\geqslant 0, \quad \text{since} \quad Re(x, s_k) \geqslant 0 \quad (k = 1, 2, \ldots). \end{aligned}$$

$\therefore x \in (c\ell S)^*$

(f) Let $x \in S + T$, i.e. $x = s + t$, $s \in S$, $t \in T$ and let $y \in S^* \cap T^*$. Then

$$\mathrm{Re}(y, x) = \mathrm{Re}(y, s) + \mathrm{Re}(y, t) \geqslant 0$$

$\therefore y \in (S + T)^*$

Conversely:

$$\begin{aligned} 0 \in S &\Rightarrow T = 0 + T \subset S + T \\ &\Rightarrow (S + T)^* \subset T^* \end{aligned}$$

Similarly

$$0 \in T \Rightarrow (S + T)^* \subset S^*$$

$\therefore 0 \in S \cap T \Rightarrow (S + T)^* \subset S^* \cap T^*$

(g) Let $x \in S^* + T^*$, i.e., $x = y + z$, $y \in S^*$, $z \in T^*$ and let $u \in S \cap T$. Then

$$\mathrm{Re}(x, u) = \mathrm{Re}(y, u) + \mathrm{Re}(z, u) \geqslant 0$$

$\therefore x \in (S \cap T)^*$ □

1.4. EXAMPLE. The converse of 1.3(g) is false even if S, T are closed convex cones. Let:

S be the set in R^3 consiting of all vectors forming an angle $\leqslant 45°$

with the vector

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

T similarly defined for

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Then:

$$S = S^*, \qquad T = T^*$$

$$S \cap T = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} : x_3 \geqslant 0 \right\}$$

$$(S \cap T)^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_3 \geqslant 0; \quad x_1 , x_2 \text{ arbitrary} \right\}$$

$$S^* + T^* = S + T \text{ does not contain } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore:

$$(S \cap T)^* \not\subset S^* + T^*.$$

See also Corollary 1.7 below.

1.5. THEOREM. *Let S be a subset of C^n. Then S is a closed convex cone iff $S = S^{**}$.*

PROOF.

If: By 1.3(a).

Only if: Let S be a closed convex cone. By 1.3(c) enough to show $S^{**} \subset S$. Suppose $x \notin S$. Then by Corollary 0.3.5, there is a $u \in S^*$ such that

$$\operatorname{Re}(u, x) < 0.$$

$\therefore\ x \notin S^{**}$. □

1.6. COROLLARY. *Let $S \subset C^n$ be a convex cone. Then*

$$c\ell S = S^{**}.$$

PROOF. Follows from Theorems 1.3(e) and 1.5. □

1.7. Corollary. *Let S, T be closed convex cones in C^n. Then*:

$$(S \cap T)^* = c\ell(S^* + T^*).$$

Proof.

$$\begin{aligned}(S \cap T)^* &= (S^{**} \cap T^{**})^*, &&\text{by } 1.5\\ &= (S^* + T^*)^{**}, &&\text{by } 1.3(\text{f})\\ &= c\ell(S^* + T^*), &&\text{by } 1.3(\text{a}) \text{ and } 1.6. \qquad \square\end{aligned}$$

For closed convex cones and polar sets in R^n, the reader is referred to [17], [18], [19], [20], and [37].

2. The Solvability Theory

2.1. Definition. Let $A \in C^{m\times n}$, $b \in C^m$ and let S be a closed convex cone in C^n. The system:

$$Ax = b, \qquad x \in S \tag{1}$$

is

(i) *consistent* if there is an x satisfying (1)

(ii) *a symptotically consistent* if there is a sequence $\{x_k : k = 1, 2,...\} \subset S$ such that $\lim_{k\to\infty} Ax_k = b$.

2.2. Theorem. *Let $A \in C^{m\times n}$, $b \in C^m$ and let $S \subset C^n$ be a nonempty closed convex cone. Then the following are equivalent*:

(a) *The system*

$$(1) \quad Ax = b, \qquad x \in S$$

is asymptotically consistent.

(b) $A^H y \in S^* \Rightarrow \operatorname{Re}(b, y) \geqslant 0$

(c) $b \in R(A)$ *and* $A^+b \in c\ell(N(A) + S)$.

Proof.

(a) $\Rightarrow$ (b):

Let

$$\{x_k : k = 1, 2,...\} \subset S, \lim_{k\to\infty} Ax_k = b.$$

Then:

$$\begin{aligned}Re(b, y) &= \lim_{k\to\infty} Re(Ax_k, y) = \lim_{k\to\infty} Re(x_k, A^H y)\\ &\geqslant 0 \quad \text{if} \quad A^H y \in S^* \quad \text{since} \quad x_k \in S, \qquad (k = 1, 2,...).\end{aligned}$$

(b) $\Rightarrow$ (c):

First we show that $b \in R(A)$. For suppose

$$P_{N(A^H)}b \neq 0$$

then

$$A^H(-P_{N(A^H)}b) = 0 \in S^*,$$

but

$$\operatorname{Re}(b, -P_{N(A^H)}b) = -\| P_{N(A^H)}b \|^2 < 0, \quad \text{contradicting (b).}$$

Next we prove that $A^+b \in c\ell(N(A) + S)$:

For any $y \in C^m$ such that $A^H y \in S^*$

$$\begin{aligned} \operatorname{Re}(b, y) &= \operatorname{Re}(AA^+b, y), \quad \text{since } b \in R(A) \text{ and } AA^+ = P_{R(A)}, \text{ e.g., [3]} \\ &= \operatorname{Re}(A^+b, A^H y) \\ &\geqslant 0, \quad \text{by } (b). \end{aligned}$$

$$\begin{aligned} \therefore A^+b &\in (R(A^H) \cap S^*)^* \\ &= c\ell(R(A^H)^* + S^{**}), \quad \text{by } 1.7 \\ &= c\ell(N(A) + S), \quad \text{by } 1.5 \text{ and } 1.2(a). \end{aligned}$$

(c) $\Rightarrow$ (a):

Let

$$A^+b \in c\ell(N(A) + S),$$

i.e.,

$$A^+b = \lim_{k\to\infty} (y_k + x_k) \quad \text{where} \quad \{x_k\} \subset S, \quad \{y_k\} \subset N(A).$$

Then

$$\begin{aligned} b &= AA^+b, \quad \text{since } b \in R(A) \\ &= A \lim_{k\to\infty} (y_k + x_k) \\ &= \lim_{k\to\infty} Ax_k, \quad \text{since } \{y_k\} \subset N(A). \end{aligned}$$ □

The equivalence of 2.2(a) and 2.2(b) was proved in [4] for the real case.

2.3. Lemma. *Let A, b, S be as in Theorem 2.2. Then the following are equivalent*:

(a) *The system*

$$Ax = b, \quad x \in S \tag{1}$$

is consistent.

(b) $b \in R(A)$ *and* $A^+b \in N(A) + S$.

Proof. The set $\{x : Ax = b\}$ is nonempty iff $b \in R(A)$, in which case

$\{x : Ax = b\} = A^+b + N(A)$. Thus (1) is solvable iff $b \in R(A)$ and $\{A^+b + N(A)\} \cap S \neq \varnothing$. The latter is equivalent to $A^+b \in N(A) + S$. □

2.4. THEOREM. *Let A, b, S be as in Theorem 2.2 and let $N(A) + S$ be closed. Then the following are equivalent*:

(a) *The system*

$$Ax = b, \qquad x \in S \tag{1}$$

is consistent.

(b) $A^H y \in S^* \Rightarrow \operatorname{Re}(b, y) \geqslant 0$.

PROOF.

(a) ⇒ (b):

From 2.2(a) ⇒ 2.2(b).

(b) ⇒ (a):

(b) implies, by 2.2 and $N(A) + S$ being closed, that

$$b \in R(A) \qquad \text{and} \qquad A^+b \in N(A) + S.$$

which, by 2.3, implies (a). □

2.5. EXAMPLE. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and S the cone of Example 1.4, i.e.,

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 : 2x_1x_3 \geqslant x_2^2, \qquad x_1 \geqslant 0 \right\}. \qquad \text{(check!)}$$

Then:

(i) 2.4(b) holds:

$$A^T y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix} \in S^* = S$$

$$\Rightarrow y_1 = 0 \Rightarrow (b, y) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = 0.$$

(ii) 2.4(a) is not satisfied:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x_2 = 1, \qquad x_3 = 0$$

but

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S, \qquad x_2 = 1 \Rightarrow x_3 > 0.$$

(iii) 2.2(a) holds, i.e. the system

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S$$

is asymptotically consistent. Indeed, the sequence:

$$x_k = \begin{pmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \end{pmatrix} = \begin{pmatrix} k \\ 1 \\ \dfrac{1}{2k} \end{pmatrix} \qquad (k = 1, 2, \ldots,)$$

satisfies $x_k \in S$, $\lim_{k\to\infty} Ax_k = b$.

In this example $N(A) + S$ is not closed (check!), therefore 2.2 is applicable, but not 2.4.

3. Polyhedral Systems

Comparing 2.4 and 2.2 we observe that their common part (b) is equivalent to

$$Ax = b, \qquad x \in S \tag{1}$$

being consistent, or asymptotically consistent, according to the cone: $N(A) + S$ being closed or not.

In this section we study a class of systems (1) for which $N(A) + S$ is always closed.

3.1. Definitions.

(a) A set $S \subset C^n$ is a *polyhedral (convex) cone* if for some integer $k > 0$ there is a $B \in C^{n\times k}$ such that

$$S = BR_+{}^k = \{Bx : x \in R_+{}^k\}.$$

(b) The system (1) is *polyhedral* if S is a polyhedral cone.

3.2. EXAMPLES.

(a) A real finite-dimensional space is a polyhedral cone: Let $\{\ell_1, \ell_2, \ldots, \ell_p\}$ be a basis of the space L and let

$$\ell_{p+1} = -\sum_{i=1}^{p} \ell_i .$$

Then

$$L = \left\{\sum_{i=1}^{p+1} \ell_i x_i : x_i \geqslant 0 \qquad (i = 1, \ldots, p+1)\right\}.$$

(b) A complex finite-dimensional space is a polyhedral cone.

(c) If S and T are polyhedral cones then so is: $S + T$.

(d) If $S_j \subset C^{n(j)}$ is a polyhedral cone, $(j = 1, \ldots, k)$ then so is

$$S_1 \times S_2 \times \cdots \times S_k \subset C^{\Sigma n(j)}$$

(e) T_α, of 1.2(e), is a polyhedral cone.

3.3. REMARK. A *polyhedral (convex) set* is usually defined as the nonempty intersection of finitely many closed halfspaces. A *polyhedral (convex) cone* is then defined as a polyhedral set which is a cone (i.e., the nonempty intersection of finitely many closed halfspaces, each having 0 in its boundary).

The fundamental Minkowski–Farkas–Weyl theorem ([38], see also [10], [17], [19], [20] and [37]) states that the latter definition is equivalent to 3.1(a).

Other characterizations of polyhedral convex cones are given in [31] and [39].

For our purposes Definition 3.1(a) suffices, and the equivalent characterization in terms of closed halfspaces is not used. The only property of polyhedral cones needed here is:

3.4. LEMMA. *Let S be a polyhedral cone in C^n. Then S is closed.*

PROOF. Let $B \in C^{n\times k}$ satisfy 3.1(a), and let $\{y_p : p = 1, 2, \ldots\} \subset S$ be a convergent sequence: $y_p \rightarrow y$.
Now $y_p \in S = BR_+{}^k$ means that $y_p = Bx_p$ for some $x_p \in R_+{}^k$

$\therefore \quad x_p \in B^+ y_p + N(B)$

$\therefore \quad x_p = B^+ y_p + z_p$ where $z_p \in N(B)$,

$$\mathrm{Im}(z_p) = -\mathrm{Im}(B^+ y_p) \qquad \text{since} \quad x_p \text{ is real,}$$

and

$$\mathrm{Re}(z_p) \geqq -\mathrm{Re}(B^+ y_p) \qquad \text{since} \quad x_p \geqq 0 \qquad (p = 1, 2, \ldots).$$

Let:

$$X_p = \{B^+y_p + z : z \in N(B), \operatorname{Im}(z) = -\operatorname{Im}(B^+y_p), \operatorname{Re}(z) \geqq -\operatorname{Re}(B^+y_p)\}$$

X_p is a nonempty closed convex set for $p = 1, 2, \ldots$.

For any $x \in X_p : Bx = Bx_p = y_p (p = 1, 2, \ldots)$. Let $\tilde{x}_p$ be the point in X_p closest to 0, $(p = 1, 2, \ldots)$. From $y_p \to y$ it follows that $B^+y_p \to B^+y$ and that the sequence $\{\tilde{x}_p : p = 1, 2, \ldots\}$ is bounded (why?). Let $\{\tilde{x}_{p(j)} : j = 1, 2, \ldots\}$ be a convergent subsequence of $\{\tilde{x}_p : p = 1, 2, \ldots\}$, say $\lim_{j\to\infty} \tilde{x}_{p(j)} = \tilde{x}$. Then $\tilde{x} \in R_+{}^k$ (why?) and

$$\begin{aligned} y &= \lim_{p\to\infty} y_p = \lim_{p\to\infty} Bx_p \\ &= \lim_{p\to\infty} B\tilde{x}_p = \lim_{j\to\infty} B\tilde{x}_{p(j)} \\ &= B \lim_{j\to\infty} \tilde{x}_{p(j)} = B\tilde{x} \in S. \end{aligned}$$

Therefore S is closed. □

3.5. Theorem. *Let $A \in C^{m\times n}$, $b \in C^m$ and let S be a polyhedral cone in C^n. Then the following are equivalent*:

(a) The system:

$$Ax = b, \qquad x \in S \tag{1}$$

is consistent.

(b) $A^H y \in S^* \Rightarrow \operatorname{Re}(b, y) \geqslant 0$.

Proof. By 2.4 enough to show that $N(A) + S$ is closed. Now:

$N(A)$ is a polyhedral cone, by 3.2(b)

$\therefore N(A) + S$ is a polyhedral cone, by 3.2(c)

$\therefore N(A) + S$ is closed, by 3.4. □

The remainder of this paper consists of applications of theorem 3.5, to special polyhedral systems (corollaries 3.6-3.8) and to complex linear programming (Section 4).

3.6. Corollary. *Let $A \in C^{m\times n}$, $b \in C^m$. Then the following are equivalent*:

(a) *The system*

$$Ax = b$$

is consistent.

(b) $A^H y = 0 \Rightarrow (b, y) = 0$.

PROOF. Using Theorem 3.5 with $S = C^n$ (a polyhedral cone by 3.2(b)) and $S^* = \{0\}$ (by 1.2(a)) we get that 3.6(a) is equivalent to:

(b′) $A^H y = 0 \Rightarrow \operatorname{Re}(b, y) \geqslant 0$.

We show now that (b′) is equivalent to (b). Clearly (b) $\Rightarrow$ (b′). To prove (b′) $\Rightarrow$ (b), assume (b′) true and (b) false, i.e., there is a y_1 satisfying:

$$A^H y_1 = 0 \quad \text{and} \quad \begin{cases} \quad\ \text{(i)} \ \operatorname{Re}(b, y_1) > 0 & \\ \text{or (ii)} \ \operatorname{Im}(b, y_1) > 0, & \operatorname{Re}(b, y_1) = 0 \\ \text{or (iii)} \ \operatorname{Im}(b, y_1) < 0, & \operatorname{Re}(b, y_1) = 0. \end{cases}$$

Define y_2 in these cases as:

$$y_2 = \begin{cases} \text{(i)} & -y_1 \\ \text{(ii)} & iy_1 \\ \text{(iii)} & -iy_1 . \end{cases}$$

Then: $A^H y_2 = 0$ and $\operatorname{Re}(b, y_2) < 0$, contradicting (b'). □

This corollary is Theorem 1 of the introduction. Similarly we prove Theorem 2 as:

3.7. COROLLARY (Farkas [16]). *Let* $A \in R^{m \times n}$, $b \in R^m$. *Then the following are equivalent:*

(a) *The system*

$$Ax = b, \qquad x \geqq 0$$

is consistent.

(b) $A^T y \geqq 0 \Rightarrow (b, y) \geqslant 0$.

PROOF. This follows from the real version of 3.5 with $S = R_+{}^n = S^*$ □

Much of the theory of linear inequalities (references for which are [8], [14], [15], [29], [32] and their bibliographies) can be based on Farkas theorem (whose many different proofs include [2], [13], [17], [20], [25], [28], [32], [34] and [36]). Of the various extensions and generalizations of 3.7 we mention [4], [6], [7], [9], [10], [13], [14], [26] and [30]. The latter extension, to the complex case, is:

3.8 COROLLARY (Levinson, [30], Theorem 1.3).

Let $A \in C^{m \times n}$, $b \in C^m$, and let $\alpha \in R^n$ satisfy $0 \leqq \alpha \leqq (\pi/2)e$. Then the following are equivalent:

(a) The system

$$Ax = b, \qquad |\arg x| \leqq \alpha$$

is consistent.

(b) $$|\arg(A^H y)| \leqq \frac{\pi}{2} e - \alpha \Rightarrow Re(b, y) \geqslant 0.$$

PROOF. Follows from theorem 3.5 with

$$S = T_\alpha, \qquad S^* = T_{(\pi/2)e-\alpha},$$

(e.g., 1.2(e) and 3.2(e)). □

4. Duality in Complex Linear Programming

The duality theory of linear programming was extended to the complex case in [30]. (Similar extension of quadratic programming is given in [23]). In this section we use the solvability Theorem 3.5 to prove a duality theorem for complex linear programming, generalizing that of [30].

4.1. DEFINITION. Let $A \in C^{m\times n}$, $b \in C^m$, $c \in C^n$ and let S be a polyhedral cone in C^n. Consider the pair of linear programs:
(I.S) maximize $\mathrm{Re}(c, x)$
subject to

$$Ax = b, \qquad x \in S \tag{1}$$

(II.S^*) minimize $\mathrm{Re}(b, y)$
subject to

$$A^H y - c \in S^*. \tag{2}$$

The problem (I.S) [(II.S^*)] is *consistent* if so is (1) [(2)].
The solutions of (1) [(2)] are called *feasible solutions* of (I.S) [(II.S^*)].
$x_0[y_0]$ is an *optimal solution* of (I.S) [(II.S^*)] if it is a feasible solution and

$$\infty > \mathrm{Re}(c, x_0) = \max\{\mathrm{Re}(c, x) : Ax = b, x \in S\}$$

$$[-\infty < \mathrm{Re}(b, y_0) = \min\{\mathrm{Re}(b, y) : A^H y - c \in S^*\}].$$

(I.S) [(II.S^*)] is *unbounded* if it has sequences of feasible solutions $\{x_k\}[\{y_k\}]$ such that $\mathrm{Re}(c, x_k) \to \infty$ [$\mathrm{Re}(b, y_k) \to -\infty$].
In what follows (I.S) and (II.S^*) denote the problems defined above for given $A \in C^{m\times n}$, $b \in C^m$, $c \in C^n$ and a polyhedral cone S in C^n.

4.2. LEMMA. *Let* (II.S^*) *be inconsistent. Then* (I.S) *is inconsistent or unbounded.*

PROOF. Let (2) be inconsistent. Then:

$$(A^H, \quad -c)\binom{y}{y_{m+1}} \in S^*, \qquad Im(y_{m+1}) = 0 \Rightarrow Re(y_{m+1}) \leqslant 0 \quad (why\,?). \tag{3}$$

Consider now the set:

$$T = S \times R \quad \text{in} \quad C^{n+1}$$

T is a polyhedral cone, by 3.2(d)

$$T^* = S^* \times iR, \quad \text{by} \quad 1.2(c) \quad \text{and} \quad 1.2(b),$$

(3) can now be rewritten as:

$$\begin{pmatrix} A^H & -c \\ 0 & i \end{pmatrix}\begin{pmatrix} y \\ y_{m+1} \end{pmatrix} \in T^* \Rightarrow Re\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} y \\ y_{m+1} \end{pmatrix}\right) \geqslant 0$$

which, by 3.5, is equivalent to the system

$$\begin{pmatrix} A & 0 \\ -c^H & -i \end{pmatrix}\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in T \tag{4}$$

being consitent.

From (4) follows the existence of an $x_1 \in S$ satisfying

$$Ax_1 = 0, \qquad \text{Re}(c^H x_1) = 1 \qquad (\text{why?}).$$

If x_0 is any feasible solution of (I.S), then $x_0 + \alpha x_1$ is feasible for any $\alpha > 0$, and

$$\text{Re}(c, x_0 + \alpha x_1) = \text{Re}(c, x_0) + \alpha \to \infty, \qquad \text{as} \quad \alpha \to \infty.$$

Therefore, (I.S) is unbounded, if consistent. □

4.3. Lemma. *Let* (I.S) *be inconsistent. Then* (II.S^*) *is inconsistent or unbounded.*

Proof. Let (1) be inconsistent. Then

$$A^H y \in S^*, \qquad \text{Re}(b, y) < 0 \tag{5}$$

is consistent, by theorem 3.5.

Let y_1 be a solution of (5). If y_0 is any feasible solution of (II.S^*) then $y_0 + \alpha y_1$ is feasible for all $\alpha > 0$ and

$$\text{Re}(b, y_0 + \alpha y_1) \to -\infty \qquad \text{as} \quad \alpha \to \infty.$$

Therefore (II.S^*) is unbounded, if consistent. □

4.4. Lemma. *If* x_0 *and* y_0 *are feasible solutions of* (I.S) *and* (II.S^*), *respectively, then*:

$$\text{Re}(c, x_0) \leqslant \text{Re}(b, y_0).$$

PROOF.

$$\begin{aligned}
\operatorname{Re}(b, y_0) &= \operatorname{Re}(Ax_0, y_0)\\
&= \operatorname{Re}(x_0, A^H y_0)\\
&= \operatorname{Re}(x_0, c + f), \qquad \text{where} \quad f = A^H y_0 - c \in S^*\\
&= \operatorname{Re}(x_0, c) + \operatorname{Re}(x_0, f)\\
&\geqslant \operatorname{Re}(x_0, c), \qquad \text{since} \quad x_0 \in S, \quad f \in S^*.
\end{aligned}$$ □

4.5. LEMMA. *If* (I.S) *and* (II.S^*) *are consistent then* (I.S) *and* (II.S^*) *have optimal solutions, and*

$$\max\{\operatorname{Re}(c, x) : Ax = b, x \in S\} = \min\{\operatorname{Re}(b, y) : A^H y - c \in S^*\} \tag{6}$$

PROOF. Let x_0 and y_0 be feasible solutions of (I.S) and (II.S^*), respectively. If

$$\operatorname{Re}(c, x_0) = \operatorname{Re}(b, y_0)$$

then the proof is completed by 4.4.

Therefore let:

$$\begin{aligned}
\alpha_0 &= \operatorname{Re}(b, y_0) - \operatorname{Re}(c, x_0) > 0 \qquad \text{(by 4.4)}\\
\beta_0 &= \operatorname{Im}(b, y_0) - \operatorname{Im}(c, x_0).
\end{aligned}$$

For any two real numbers (α, β) consider now the pair of problems

$$\begin{pmatrix} -c^H & b^H & 0 & -i\\ A & 0 & 0 & 0\\ 0 & -A^H & I & 0 \end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ t \end{pmatrix} = \begin{pmatrix} \alpha + i\beta\\ b\\ -c \end{pmatrix}, \quad \begin{pmatrix} x\\ y\\ z\\ t \end{pmatrix} \in K = S \times C^m \times S^* \times R. \tag{7. α, β}$$

$$\begin{pmatrix} -c & A^H & 0\\ b & 0 & -A\\ 0 & 0 & I\\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} u\\ v\\ w \end{pmatrix} \in K^* = S^* \times \{0\} \times S \times iR, \; Re\left(\begin{pmatrix} \alpha + i\beta\\ b\\ -c \end{pmatrix}, \begin{pmatrix} u\\ v\\ w \end{pmatrix}\right) < 0, \tag{8. α, β}$$

By Theorem 3.5 the problem (7. α, β) is inconsistent iff the problem (8. α, β) has solutions.

Let (6) be false. Then (7. $0, \beta$) is inconsistent for all real β, and (8. $0, \beta$) is accordingly consistent for all real β.

Let $\begin{pmatrix} u\\ v\\ w \end{pmatrix}$ be any solution of (8. $0, \beta_0$). Since u is real ($iu \in iR$) we distinguish two cases:

$u \leqslant 0$: Then $\begin{pmatrix} v \\ v \\ w \end{pmatrix}$ is also a solution of (8. α, β_0) for all $\alpha > 0$ (why ?), and in particular of (8. α_0, β_0). Therefore (7. α_0, β_0) is inconsistent.

But $\begin{pmatrix} x_0 \\ y_0 \\ A^H y_0 - c \\ 0 \end{pmatrix}$ is a solution of (7. α_0, β_0), a contradiction.

$u > 0$: Assume $u = 1$. Then (8. 0, β_0) gives:

$$-c + A^H v_0 \in S^*$$
$$b - Aw_0 = 0, \qquad w_0 \in S$$
$$\mathrm{Re}(b, v_0) - \mathrm{Re}(c, w_0) < 0, \qquad \text{a contradiction to 4.4.} \qquad \square$$

Collecting the above results we get the following duality theorem for complex linear programming. To be precise, Lemmas 4.2–4.5 exclude all but the four cases of theorem 4.6. That these four cases are possible can then be demonstrated by simple (real) examples.

4.6. THEOREM. *Let $A \in C^{m\times n}$, $b \in C^m$, $c \in C^n$ and S be a polyhedral cone in C^n.*

Consider the problems:

(I.S) maximize $\mathrm{Re}(c, x)$

$$\text{s.t. } Ax = b, \qquad x \in S$$

(II.S^*) minimize $\mathrm{Re}(b, y)$

$$\text{s.t. } A^H y - c \in S^*.$$

Then exactly one of the following four cases hold:

(a) Both (I.S) and (II.S^*) are consistent, have optimal solutions, and $\max\{\mathrm{Re}(c, x) : Ax = b, x \in S\} = \min\{\mathrm{Re}(b, y) : A^H y - c \in S^*\}$

(b) (I.S) inconsistent, (II.S^*) unbounded

(c) (I.S) unbounded, (II.S^*) inconsistent

(d) Both (I.S) and (II.S^*) are inconsistent. $\square$

4.7. REMARKS.

(a) For $S = T_\alpha$ of 1.2(e), theorem 4.6 restates the duality theorem of [30].

(b) If S is a general (non polyhedral) convex cone then the duality situation becomes more complicated. Then, instead of the four cases of 4.6, there are ten possible cases, e.g. [5] for the real case.

(c) In the real case, Theorem 4.6 gives for $S = R_+{}^n = S^*$ the classical duality theorem of linear programming.

(d) The proofs of this section are greatly simplified when restricted to the real case. Thus for example instead of the problems (7. α, β) and (8. α, β) we consider simpler problems (7. α) and (8. α) where:

$$\begin{pmatrix} -c^T & b^T & 0 \\ A & 0 & 0 \\ 0 & -A^T & I \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ b \\ -c \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S \times C^m \times S^*, \tag{7.α}$$

etc.

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